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Structural and sparsity properties of symmetric 7-matrices ☆

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ABSTRACT

A matrix $A = (a_{ij})$ is called a $7_{\alpha,\beta}$-matrix if its entries satisfy the recurrence relation $\alpha a_{i-1,j-1} + \beta a_{i-1,j} = a_{ij}$, where α and β are fixed nonzero real numbers. In this paper, we study the structural and sparsity properties of symmetric $7_{\alpha,\beta}$-matrices. In particular, we determine the largest number of zero entries of a nonzero $7_{\alpha,\beta}$-matrix and determine the matrices that attain this largest number.

1. Introduction

Matrices whose entries satisfy some specified relation have been objects of interest for years, such as magic squares [1] and Pascal matrices [2]. In [3], the authors introduced a class of matrices $A = (a_{ij})$ whose entries satisfy the recurrence relation

$$\alpha a_{i-1,j-1} + \beta a_{i-1,j} = a_{ij}, \tag{1.1}$$

where α and β are a pair of nonzero real numbers. Such matrices are said to be $7_{\alpha,\beta}$-matrices since the relative positions of the entries $a_{i-1,j-1}$, $a_{i-1,j}$ and a_{ij} in (1.1) form the "mirrored gamma" or "figure 7" shape. The relation (1.1) is called the $7_{\alpha,\beta}$-law and we call a matrix a 7-matrix if it is a $7_{\alpha,\beta}$-matrix for some α, β. For example, lower triangular Pascal matrices are $7_{1,1}$-matrices. It is easily seen from the definition that a 7-matrix is completely determined by its first row, first column and the

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recurrence relation. The structure and some properties along with some applications of 7-matrices were investigated in [3]. The motivational probability model for studying such interesting matrices was also described in [3].

It seems that 7-matrices do not have nice analytic properties. In this paper, we consider a special class of 7-matrices, the symmetric 7-matrices. Compared with general 7-matrices, symmetric 7-matrices have a simple structure and some nice properties.

Throughout this paper, all matrices are square and real. The set of all $7_{\alpha,\beta}$-matrices of order n shall be denoted by $M_n(\alpha, \beta)$ whereas $S_n(\alpha, \beta)$ denote the set of all $n \times n$ symmetric $7_{\alpha,\beta}$-matrices. We denote by I_n the identity matrix of order n. We refer to the k-th skew-diagonal of a matrix $A = (a_{ij})$ as the entries whose subscripts satisfy $i + j = k + 1$. If all of the entries on the k-th skew-diagonal are equal, then we say that A has a constant k-th skew-diagonal. Given a pair of nonzero real numbers α and β, we call a vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ $(n \geqslant 3)$ an (α, β)-vector if $x_i = \alpha x_{i-2} + \beta x_{i-1}$ for every i, $3 \leqslant i \leqslant n$. An (α, β)-vector is completely determined by its first two components and the recurrence relation. Let $V_n(\alpha, \beta)$ denote the set of all (α, β)-vectors with n components.

Theorem 1. *For $n \geqslant 3$, $V_n(\alpha, \beta)$ is a 2-dimensional vector space.*

Proof. It is easy to check that $V_n(\alpha, \beta)$ is a subspace of $\mathbb{R}^n$. Observe that a nonzero vector $x \in \mathbb{R}^n$ is an (α, β)-vector if and only if x is an eigenvector of the following lower triangular matrix

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ \alpha & \beta & 0 & 0 & \cdots & 0 \\ 0 & \alpha & \beta & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha & \beta & 0 \end{pmatrix}_{n\times n} \tag{1.2}$$

corresponding to the eigenvalue 1.

Thus, $V_n(\alpha, \beta) = V_1$, where $V_1 = \{x | Lx = x\} = \ker(L - I_n)$. It is easy to see that $\operatorname{rank}(L - I_n) = n - 2$. Hence, $\dim V_n(\alpha, \beta) = \dim V_1 = 2$. $\square$

2. Structure and properties of symmetric 7-matrices

In this section, we will determine the structure and investigate some properties of symmetric 7-matrices. For a matrix $A \in M_n(\alpha, \beta)$, we denote the first row and the first column of A by $r(A)$ and $c(A)$, respectively.

Lemma 2. *Let $A \in M_n(\alpha, \beta)$, $n \geqslant 3$. Then $A \in S_n(\alpha, \beta)$ if and only if $r(A) = c(A)^T$ and $r(A)$ is an (α, β)-vector.*

Proof. Let $r(A) = (x_1, x_2, \ldots, x_n)$ and $c(A) = (x_1, y_2, \ldots, y_n)^T$. If $A = (a_{ij}) \in M_n(\alpha, \beta)$ is symmetric, then it is clear that $r(A) = c(A)^T$. It is easily seen that $a_{22} = \alpha a_{11} + \beta a_{12} = \alpha x_1 + \beta x_2$, $a_{23} = \alpha a_{12} + \beta a_{13} = \alpha x_2 + \beta x_3$ and $a_{32} = \alpha a_{21} + \beta a_{22} = \alpha x_2 + \beta(\alpha x_1 + \beta x_2)$. Hence, it follows from $a_{23} = a_{32}$ and $\beta \neq 0$ that $x_3 = \alpha x_1 + \beta x_2$. Similarly, $a_{24} = \alpha x_3 + \beta x_4$, $a_{42} = \alpha x_3 + \beta(\alpha x_2 + \beta x_3)$ and thus $x_4 = \alpha x_2 + \beta x_3$. Continuing in this way we deduce that $x_i = \alpha x_{i-2} + \beta x_{i-1}$ for every i, $i = 3, \ldots, n$, i.e., $r(A)$ is an (α, β)-vector.

Conversely, if $r(A) = c(A)^T$ and $r(A)$ is an (α, β)-vector, then A is of the form

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} & x_n \\ x_2 & x_3 & \cdots & x_n & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1} & x_n & \cdots & * & * \\ x_n & * & \cdots & * & * \end{pmatrix}.$$

That is, A has constant k-th skew-diagonals, $k = 1, \ldots, n$. Using the recurrence relation (1.1), it follows that A has a constant $(n+1)$-th skew-diagonal, each entry on which is $\alpha x_{n-1} + \beta x_n$. Continuing in this way we conclude that A has constant k-th skew-diagonals for every k, $k = 1, \ldots, 2n-1$, which implies the symmetry of A. $\square$

A Hankel matrix

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_2 & c_3 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{pmatrix}$$

is a square matrix in which for every r, $1 \leqslant r \leqslant 2n-1$, the entries on the r-th skew-diagonal are the same. It can be seen from Lemma 2 that if $A \in S_n(\alpha, \beta)$, then each row (resp. each column) of A is an (α, β)-vector. Let $x_i = \alpha x_{i-2} + \beta x_{i-1}$, $3 \leqslant i \leqslant 2n-1$. Then a symmetric 7-matrix is of the form

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_3 & \cdots & x_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_{n+1} & \cdots & x_{2n-1} \end{pmatrix}, \tag{2.1}$$

which is a Hankel matrix and is completely determined by the first two components of its first row or first column, i.e., x_1 and x_2.

Theorem 3. *For $n \geqslant 3$, $S_n(\alpha, \beta)$ is a 2-dimensional vector space.*

Proof. Let $A, B \in S_n(\alpha, \beta)$ and let $\lambda, \mu \in \mathbb{R}$. Let $C = \lambda A + \mu B$. A simple computation shows that the entries of C satisfy the $7_{\alpha,\beta}$-law (1.1) and $C = C^T$, i.e., $C \in S_n(\alpha, \beta)$, so that $S_n(\alpha, \beta)$ is a subspace of the vector space of all $n \times n$ real matrices.

By the structure of the matrix in (2.1), it is easily seen that $S_n(\alpha, \beta)$ is isomorphic to $V_n(\alpha, \beta)$. Hence, by Theorem 1, dim $S_n(\alpha, \beta)$=dim $V_n(\alpha, \beta) = 2$. $\square$

Theorem 4. *Let $A \in S_n(\alpha, \beta)$ be of the form (2.1), $n \geqslant 3$. Then*
(1) $LA = AL^T = A$, where L is defined in (1.2);
(2) span$\{w_1, w_2, \ldots, w_{n-2}\} \subseteq \ker A$, where $w_1 = (-\alpha, -\beta, 1, 0, \ldots, 0)^T$, $w_2 = (0, -\alpha, -\beta, 1, 0, \ldots, 0)^T$, ..., $w_{n-2} = (0, \ldots, 0, -\alpha, -\beta, 1)^T$;
(3) A is congruent to $\begin{pmatrix} x_1 & x_2 \\ x_2 & \alpha x_1 + \beta x_2 \end{pmatrix} \oplus \mathbf{0}_{n-2}$.

Proof

(1) Since each row and each column of A is an (α, β)-vector, we have $LA = AL^T = A$.

(2) By a direct computation we see that $Aw_i = 0, i = 1, \ldots, n-2$. Hence, any linear combination of $w_1, \ldots, w_{n-2}$ belongs to the kernel of A.

(3) Let

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ -\alpha & -\beta & 1 & 0 & \cdots & 0 \\ 0 & -\alpha & -\beta & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -\alpha & -\beta & 1 \end{pmatrix}_{n\times n}. \tag{2.2}$$

Then C is invertible and $CAC^T = \begin{pmatrix} x_1 & x_2 \\ x_2 & \alpha x_1 + \beta x_2 \end{pmatrix} \oplus \mathbf{0}_{n-2}$. $\square$

We write $A \geqslant 0$ if A is positive semidefinite, which means that $\langle Ax, x\rangle \geqslant 0$ for all $x \in \mathbb{R}^n$.

Corollary 5. *Let $A \in S_n(\alpha, \beta)$ be of the form (2.1), $n \geqslant 2$. Then $\operatorname{rank}(A) \leqslant 2$ and $A \geqslant 0$ if and only if $\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \geqslant 0$.*

For two matrices $A, B \in S_n(\alpha, \beta)$, AB is not symmetric in general. The following theorem gives a sufficient and necessary condition under which AB is symmetric, or equivalently, A and B commute.

For the sake of convenience, if $A \in S_n(\alpha, \beta)$ is determined by the two real numbers x_1 and x_2, then we write $A = [x_1, x_2]_{\alpha,\beta}$.

Theorem 6. *Let $A = [x_1, x_2]_{\alpha,\beta}, B = [y_1, y_2]_{\alpha,\beta} \in S_n(\alpha, \beta)$, $n \geqslant 2$. If n is even, then AB is symmetric if and only if $x_1y_2 = x_2y_1$ or $\alpha = 1$. If n is odd, then AB is symmetric if and only if $x_1y_2 = x_2y_1$.*

Proof. For $A = [x_1, x_2]_{\alpha,\beta}, B = [y_1, y_2]_{\alpha,\beta} \in S_n(\alpha, \beta)$, it follows from (3) of Theorem 4 that $CAC^T = X_2 \oplus \mathbf{0}_{n-2}$ and $CBC^T = Y_2 \oplus \mathbf{0}_{n-2}$, where $X_2 = \begin{pmatrix} x_1 & x_2 \\ x_2 & \alpha x_1 + \beta x_2 \end{pmatrix}$ and $Y_2 = \begin{pmatrix} y_1 & y_2 \\ y_2 & \alpha y_1 + \beta y_2 \end{pmatrix}$. A direct computation shows that

$$C(AB - BA)C^T = [X_2 \oplus \mathbf{0}_{n-2}](CC^T)^{-1}[Y_2 \oplus \mathbf{0}_{n-2}] - [Y_2 \oplus \mathbf{0}_{n-2}](CC^T)^{-1}[X_2 \oplus \mathbf{0}_{n-2}]. \tag{2.3}$$

Hence, in view of (2.3), to compute $C(AB-BA)C^T$, we need only to compute the 2×2 leading principal submatrix of $(CC^T)^{-1}$.

Observe that the first two columns of C^{-1} are (α, β)-vectors determined by 1, 0 and 0, 1, respectively. Let the first two columns of C^{-1} be z_1 and z_2. Then the 2×2 leading principal submatrix of $(CC^T)^{-1}$ is $\begin{pmatrix} z_1^Tz_1 & z_1^Tz_2 \\ z_1^Tz_2 & z_2^Tz_2 \end{pmatrix}$. Thus, the righthand side of the equality (2.3) equals $Z_2 \oplus \mathbf{0}_{n-2}$ for some 2×2 matrix Z_2. Since $C(AB - BA)C^T$ is skew-symmetric, we need only to compute $(Z_2)_{12}$, the (1,2)-position entry of Z_2. By a direct computation we have

$$(Z_2)_{12} = (z_1^Tz_1 + \beta z_1^Tz_2 - \alpha z_2^Tz_2)(x_1y_2 - x_2y_1).$$

Next, we will proceed by induction on n to show that

$$z_1^T z_1 + \beta z_1^T z_2 - \alpha z_2^T z_2 = P_n(\alpha), \tag{2.4}$$

where $P_n(\alpha) = (-\alpha)^{n-1} + (-\alpha)^{n-2} + \cdots + (-\alpha) + 1$.

It is clear that (2.4) holds for $n = 2$. If $n = 3$, then $z_1 = (1, 0, \alpha)^T$, $z_2 = (0, 1, \beta)^T$. Thus, $z_1^T z_1 + \beta z_1^T z_2 - \alpha z_2^T z_2 = \alpha^2 - \alpha + 1 = P_3(\alpha)$. Suppose (2.4) is true for $n-1$ and let $A, B \in S_n(\alpha, \beta)$. Partition z_1 and z_2 as $z_1 = (\tilde{z}_1^T, (z_1)_n)^T$ and $z_2 = (\tilde{z}_2^T, (z_2)_n)^T$, where $(z_i)_n$ is the n-th component of z_i, $i = 1, 2$. Then by the inductive hypothesis,

$$\begin{aligned} z_1^T z_1 + \beta z_1^T z_2 - \alpha z_2^T z_2 &= \tilde{z}_1^T \tilde{z}_1 + \beta \tilde{z}_1^T \tilde{z}_2 - \alpha \tilde{z}_2^T \tilde{z}_2 + (z_1)_n^2 + \beta (z_1)_n (z_2)_n - \alpha (z_2)_n^2 \\ &= P_{n-1}(\alpha) + (z_1)_n^2 + \beta (z_1)_n (z_2)_n - \alpha (z_2)_n^2. \end{aligned} \tag{2.5}$$

Observe that $(z_1)_n = \alpha (z_2)_{n-1}$ and $(z_2)_n = (z_1)_{n-1} + \beta (z_2)_{n-1}$. Then by recurrence,

$$(z_1)_n^2 + \beta (z_1)_n (z_2)_n - \alpha (z_2)_n^2 = (-\alpha)[(z_1)_{n-1}^2 + \beta (z_1)_{n-1} (z_2)_{n-1} - \alpha (z_2)_{n-1}^2] = \cdots = (-\alpha)^{n-1}.$$

Therefore, by (2.5), $z_1^T z_1 + \beta z_1^T z_2 - \alpha z_2^T z_2 = P_{n-1}(\alpha) + (-\alpha)^{n-1} = P_n(\alpha)$.

Now, we come to the conclusion that

$$C(AB - BA)C^T = P_n(\alpha)(x_1 y_2 - x_2 y_1) \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \mathbf{0}_{n-2} \right].$$

From the above equality we see that $AB = BA$ if and only if $P_n(\alpha)(x_1 y_2 - x_2 y_1) = 0$. Note that $P_n(\alpha) = 0$ for some real number α if and only if n is even and $\alpha = 1$. Therefore, if n is even, then AB is symmetric if and only if $x_1 y_2 = x_2 y_1$ or $\alpha = 1$. If n is odd, then AB is symmetric if and only if $x_1 y_2 = x_2 y_1$. $\square$

We remark that even if $A, B \in S_n(\alpha, \beta)$ and AB is symmetric, it may happen that $AB \notin S_n(\alpha, \beta)$. Just taking $A = B = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{pmatrix} \in S_3(1, 1)$, then $AB = A^2 = \begin{pmatrix} 6 & 9 & 15 \\ 9 & 14 & 23 \\ 15 & 23 & 38 \end{pmatrix}$ is not a 7-matrix.

For $A, B \in S_n(\alpha, \beta)$, the following theorem gives a sufficient condition for $AB \in S_n(\alpha, \beta)$ under some restrictions.

Theorem 7. *Let $A, B \in S_n(\alpha, \beta)$ with $\alpha \neq 1$. If $\operatorname{rank}(A) = 1$ and AB is symmetric, then $AB \in S_n(\alpha, \beta)$.*

Proof. Since AB is symmetric, by Theorem 6 we may assume that $(y_1, y_2) = k(x_1, x_2)$ for some nonzero real number k. If $\operatorname{rank}(A) = 1$, then $X_2 = \begin{pmatrix} x_1 & x_2 \\ x_2 & \alpha x_1 + \beta x_2 \end{pmatrix}$ is also of rank 1. Thus $X_2 = \begin{pmatrix} x_1 & l x_1 \\ l x_1 & l^2 x_1 \end{pmatrix}$ for some real number l. Assume that the 2×2 leading principal submatrix of $(CC^T)^{-1}$ is $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$. Then a direct computation shows that

$$\frac{1}{k} C(AB)C^T = [X_2 \oplus \mathbf{0}_{n-2}](CC^T)^{-1}[X_2 \oplus \mathbf{0}_{n-2}] = \begin{pmatrix} w_1 & l w_1 \\ l w_1 & l^2 w_1 \end{pmatrix}, \tag{2.6}$$

where $w_1 = x_1^2 z_1 + l x_1^2 z_2 + l x_1^2 z_2 + l^2 x_1^2 z_3$.

Since $X_2 \in S_2(\alpha, \beta)$, it follows that $(\alpha + l\beta)x_1 = l^2 x_1$. Since rank$(A) = 1$ implies $x_1 \neq 0$, we get $\alpha + l\beta = l^2$. Therefore, $\begin{pmatrix} \frac{1}{k}w_1 & \frac{l}{k}w_1 \\ \frac{l}{k}w_1 & \frac{l^2}{k}w_1 \end{pmatrix} \in S_2(\alpha, \beta)$. For a symmetric matrix A, it is not difficult to check that (3) of Theorem 4 is also a sufficient condition for $A \in S_n(\alpha, \beta)$. Hence by (2.6), $AB \in S_n(\alpha, \beta)$. $\square$

Theorem 8. *Let $A = [x_1, x_2]_{\alpha,\beta}$, $B = [y_1, y_2]_{\alpha,\beta} \in S_n(\alpha, \beta)$, $n \geqslant 3$. Then each row and each column of AB and BA is an (α, β)-vector.*

Proof. For any row of AB, say row i, $1 \leqslant i \leqslant n$, let j be a positive integer such that $1 \leqslant j \leqslant n-2$. Then $(AB)_{i,j} = \sum_{h=0}^{n-1} x_{i+h}y_{j+h}$, $(AB)_{i,j+1} = \sum_{h=0}^{n-1} x_{i+h}y_{j+1+h}$, $(AB)_{i,j+2} = \sum_{h=0}^{n-1} x_{i+h}y_{j+2+h}$. It follows that $(AB)_{i,j+2} = \sum_{h=0}^{n-1} x_{i+h}y_{j+2+h} = \sum_{h=0}^{n-1} x_{i+h}(\alpha y_{j+h} + \beta y_{j+h+1}) = \alpha \sum_{h=0}^{n-1} x_{i+h}y_{j+h} + \beta \sum_{h=0}^{n-1} x_{i+h}y_{j+h+1} = \alpha(AB)_{i,j} + \beta(AB)_{i,j+1}$, i.e., row i of AB is an (α, β)-vector. In a similar way, we conclude that each column of AB is an (α, β)-vector. Since $BA = (AB)^T$, each row and each column of BA is an (α, β)-vector. $\square$

Let $A \in S_n(\alpha, \beta)$ be of the form (2.1). Given two positive integers i and j with $1 \leqslant i < j \leqslant 2n-1$, we claim that there exist two polynomials in α and β, say $p_1(\alpha, \beta)$ and $p_2(\alpha, \beta)$, such that $x_j = p_1(\alpha, \beta)x_i + p_2(\alpha, \beta)x_{i+1}$, i.e., x_j can be written as a linear combination of x_i and x_{i+1}. Indeed, if $j = i+1$, then $x_j = 0x_i + 1x_{i+1}$. If $j - i = 2$, then $x_j = \alpha x_{j-2} + \beta x_{j-1} = \alpha x_i + \beta x_{i+1}$. If $j - i \geqslant 3$, then $x_j = \alpha x_{j-2} + \beta x_{j-1} = \alpha x_{j-2} + \beta(\alpha x_{j-3} + \beta x_{j-2}) = \alpha\beta x_{j-3} + (\alpha + \beta^2)x_{j-2} = \cdots = p_1(\alpha, \beta)x_i + p_2(\alpha, \beta)x_{i+1}$ for some polynomials $p_1(\alpha, \beta)$ and $p_2(\alpha, \beta)$. Moreover, if $j - i = k - j$ for some positive integer k, $j < k \leqslant 2n-1$, then x_k can be expressed as a linear combination of x_j and x_{j+1} with the same coefficients as above, i.e., $x_k = p_1(\alpha, \beta)x_j + p_2(\alpha, \beta)x_{j+1}$. This is because it takes the same number of steps to decompose x_j and x_k into the linear combination of x_i, x_{i+1} and x_j, x_{j+1}, respectively. The above facts will be frequently used throughout the rest of this paper.

Given a symmetric 7-matrix A, we are wondering about the submatrices of A which are also symmetric 7-matrices. The following theorem exhibits a class of submatrices of A which are also symmetric 7-matrices. It is worth pointing out that it may happen that some submatrices of A are symmetric $7_{\delta,\gamma}$-matrices with $\delta = 0$ or $\gamma = 0$, which is slightly different from the definition of a 7-matrix introduced in Section 1.

Theorem 9. *Let $A \in S_n(\alpha, \beta)$ be of the form (2.1), $n \geqslant 3$. Then any submatrix of A of the form*

$$\tilde{A} = \begin{pmatrix} x_i & x_{i+k} & \cdots & x_{i+lk} \\ x_{i+k} & x_{i+2k} & \cdots & x_{i+(l+1)k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i+lk} & x_{i+(l+1)k} & \cdots & x_{i+2lk} \end{pmatrix}_{(l+1)\times(l+1)} \tag{2.7}$$

is a symmetric 7-matrix, where i, l and k are positive integers satisfying $1 \leqslant i \leqslant 2n-3$ and $i+2lk \leqslant 2n-1$. In particular, if $k = 1$, then $\tilde{A} \in S_{l+1}(\alpha, \beta)$.

Proof. In order to prove that $\tilde{A}$ is a symmetric 7-matrix, we need to prove that there exist a pair of real numbers δ and γ such that $x_{i+(m+1)k} = \delta x_{i+(m-1)k} + \gamma x_{i+mk}$ for every m, $1 \leqslant m \leqslant 2l-1$. Here we allow $\delta = 0$ or $\gamma = 0$. For any m, $1 \leqslant m \leqslant 2l-1$, we can represent $x_{i+(m+1)k}$ and x_{i+mk} as a linear combination of $x_{i+(m-1)k}$ and $x_{i+(m-1)k+1}$, respectively,

$$x_{i+(m+1)k} = f_1(\alpha, \beta)x_{i+(m-1)k} + f_2(\alpha, \beta)x_{i+(m-1)k+1}, \tag{2.8}$$

$$x_{i+mk} = g_1(\alpha, \beta)x_{i+(m-1)k} + g_2(\alpha, \beta)x_{i+(m-1)k+1}, \tag{2.9}$$

where $f_i(\alpha, \beta)$ and $g_i(\alpha, \beta)$, $i = 1, 2$, are polynomials in α and β.

We consider the following two cases.

Case 1. $g_2(\alpha,\beta)\neq 0$.

By (2.9), we have $x_{i+(m-1)k+1}=\frac{x_{i+mk}-g_1(\alpha,\beta)x_{i+(m-1)k}}{g_2(\alpha,\beta)}$ and then by (2.8), we have

$$\begin{aligned}x_{i+(m+1)k} &= f_1(\alpha,\beta)x_{i+(m-1)k}+f_2(\alpha,\beta)\frac{x_{i+mk}-g_1(\alpha,\beta)x_{i+(m-1)k}}{g_2(\alpha,\beta)}\\ &=\left[f_1(\alpha,\beta)-\frac{f_2(\alpha,\beta)g_1(\alpha,\beta)}{g_2(\alpha,\beta)}\right]x_{i+(m-1)k}+\frac{f_2(\alpha,\beta)}{g_2(\alpha,\beta)}x_{i+mk}.\end{aligned}$$

Hence, $\tilde{A}$ is a symmetric $7_{\delta,\gamma}$-matrix with $\delta=[f_1(\alpha,\beta)-\frac{f_2(\alpha,\beta)g_1(\alpha,\beta)}{g_2(\alpha,\beta)}]$ and $\gamma=\frac{f_2(\alpha,\beta)}{g_2(\alpha,\beta)}$.

Case 2. $g_2(\alpha,\beta)=0$.

If $g_1(\alpha,\beta)=0$, by (2.9), $x_{i+mk}=0$ for every m, $1\leqslant m\leqslant 2l$. Moreover, $x_{i+mk+1}=g_1(\alpha,\beta)x_{i+(m-1)k+1}+g_2(\alpha,\beta)x_{i+(m-1)k+2}=0$. It is not difficult to see that $x_{i+mk}=x_{i+mk+1}=0$ implies $A=\mathbf{0}$. In particular, we have $\tilde{A}=\mathbf{0}$. Hence, $\tilde{A}$ is a symmetric $7_{\delta,\gamma}$-matrix, where δ and γ are two arbitrary real numbers.

If $g_1(\alpha,\beta)\neq 0$, by (2.9),

$$x_{i+mk}=g_1(\alpha,\beta)x_{i+(m-1)k}\ \text{for every}\ m,\ 1\leqslant m\leqslant 2l. \tag{2.10}$$

Thus, $x_{i+(m+1)k}=g_1(\alpha,\beta)x_{i+mk}=g_1^2(\alpha,\beta)x_{i+(m-1)k}$ for every m, $1\leqslant m\leqslant 2l-1$. If $x_{i+m_0k}=0$ for some m_0, $0\leqslant m_0\leqslant 2l$, it can be seen from (2.10) that $\tilde{A}=0$. Hence, $\tilde{A}$ is a symmetric 7-matrix. If $x_{i+m_1k}\neq 0$ for some m_1, $0\leqslant m_1\leqslant 2l$, it can be seen from (2.10) that $\tilde{A}$ has no zero entries. Since the equation $\delta+\gamma g_1(\alpha,\beta)=g_1^2(\alpha,\beta)$ with variables δ and γ always has real solutions, for any m, $1\leqslant m\leqslant 2l-1$, each pair of real numbers δ and γ with $\delta+\gamma g_1(\alpha,\beta)=g_1^2(\alpha,\beta)$ will satisfy $x_{i+(m+1)k}=\delta x_{i+(m-1)k}+\gamma x_{i+mk}$.

Now, we come to the conclusion that there exist real numbers δ and γ such that $x_{i+(m+1)k}=\delta x_{i+(m-1)k}+\gamma x_{i+mk}$ for every m, $1\leqslant m\leqslant 2l-1$. That is, $\tilde{A}$ is a symmetric 7-matrix. $\square$

For a matrix $A\in M_n(\alpha,\beta)$, let μ,ν be nonempty ordered subsets of $\{1,\ldots,n\}$. Denote by $A[\mu|\nu]$ the submatrix of A with rows indexed by μ and columns indexed by ν. If $\mu=\nu$, then $A[\mu|\mu]$ is abbreviated to $A[\mu]$.

Example 10. Consider the following matrix

$$A=\begin{pmatrix} 1 & 1 & 0 & -2 & -4 & -4 & 0 & 8 & 16\\ 1 & 0 & -2 & -4 & -4 & 0 & 8 & 16 & 16\\ 0 & -2 & -4 & -4 & 0 & 8 & 16 & 16 & 0\\ -2 & -4 & -4 & 0 & 8 & 16 & 16 & 0 & -32\\ -4 & -4 & 0 & 8 & 16 & 16 & 0 & -32 & -64\\ -4 & 0 & 8 & 16 & 16 & 0 & -32 & -64 & -64\\ 0 & 8 & 16 & 16 & 0 & -32 & -64 & -64 & 0\\ 8 & 16 & 16 & 0 & -32 & -64 & -64 & 0 & 128\\ 16 & 16 & 0 & -32 & -64 & -64 & 0 & 128 & 256 \end{pmatrix}\in S_9(-2,2)$$

and the 3×3 submatrix of A

$$A[\{1,5,9\}] = \begin{pmatrix} 1 & -4 & 16 \\ -4 & 16 & -64 \\ 16 & -64 & 256 \end{pmatrix}.$$

Taking $i = 1$ and $k = 4$ in Theorem 9, then a direct computation shows that $g_1(\alpha, \beta) = -4$ and $g_2(\alpha, \beta) = 0$. Hence $A[\{1, 5, 9\}] \in S_3(\delta, \gamma)$, where δ and γ satisfy $\delta - 4\gamma = 16$.

Example 11. Consider the following matrix

$$B = \begin{pmatrix} 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 \end{pmatrix} \in S_9(-1, -1)$$

and the 3×3 submatrix

$$B[\{1,5,9\}] = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Then $g_2(\alpha, \beta) = 1 \neq 0$, $g_1(\alpha, \beta) = 0$, $f_1(\alpha, \beta) = -1$, $f_2(\alpha, \beta) = -1$. Hence, by Theorem 9, $B[\{1, 5, 9\}] \in S_3(-1, -1)$.

Let us remark that if $A \in M_n(\alpha, \beta)$, $n \geqslant 3$, then some submatrices of A of the form (2.7) may not be a 7-matrix. A simple example is provided by the following matrix

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 8 & 3 & 5 & 7 & 9 & 11 & 13 \\ 9 & 11 & 8 & 12 & 16 & 20 & 24 \\ 10 & 20 & 19 & 20 & 28 & 36 & 44 \\ 11 & 30 & 39 & 39 & 48 & 64 & 80 \\ 12 & 41 & 69 & 78 & 87 & 112 & 144 \\ 13 & 53 & 110 & 147 & 165 & 199 & 256 \end{pmatrix} \in M_7(1, 1).$$

However, the 3×3 submatrix

$$C[\{1,4,7\}] = \begin{pmatrix} 1 & 4 & 7 \\ 10 & 20 & 44 \\ 13 & 147 & 256 \end{pmatrix}$$

is not a 7-matrix.

3. Sparsity of symmetric 7-matrices

For a matrix $A \in S_n(\alpha, \beta)$, let $\phi(A)$ denote the number of zeros of A. In this section, we will determine

$$\max\{\phi(A) \mid \mathbf{0} \neq A \in S_n(\alpha, \beta)\}$$

and determine the matrices that attain the maximum.

Given two positive integers i and j, $1 \leqslant i, j \leqslant 2n-1$, let us denote $[x_i]_j = \{x_{i+kj} | k = 0, 1, \ldots, \lfloor \frac{2n-1-i}{j} \rfloor\} \subseteq \{x_1, x_2, \ldots, x_{2n-1}\}$, i.e., the subscripts of the elements of $[x_i]_j$ form an arithmetic progression with first term i and common difference j. Let $\phi(x_i)$ be the number of x_i's in A and let $\phi([x_i]_j) = \sum_{x_k \in [x_i]_j} \phi(x_k)$. Denote by $|S|$ the cardinality of a set S.

Lemma 12. *Let $A \in S_n(\alpha, \beta)$ be a nonzero symmetric 7-matrix of the form (2.1). Then*

$$\phi([x_i]_j) \geqslant \phi([x_{i+1}]_j) \text{ for each pair of integers } i, j,\ 1 \leqslant j \leqslant i \leqslant 2n-1 \tag{3.1}$$

and

$$\phi([x_i]_j) > \phi([x_{i+1}]_j) \text{ for each pair of integers } i, j,\ 2 \leqslant 2j \leqslant i \leqslant 2n-1. \tag{3.2}$$

Proof. Observe that

$$\phi(x_i) = \begin{cases} i, & \text{if } 1 \leqslant i \leqslant n, \\ 2n-i, & \text{if } n \leqslant i \leqslant 2n-1. \end{cases}$$

Thus, if $n \leqslant i \leqslant 2n-1$, then (3.1) and (3.2) hold obviously.

If $1 \leqslant i \leqslant n-1$, let us partition $[x_i]_j$ and $[x_{i+1}]_j$ respectively as

$$[x_i]_j = S_1 \cup S_2, \quad [x_{i+1}]_j = S_1{}' \cup S_2{}',$$

where $S_1 = \{x_k | i \leqslant k \leqslant n\}, S_1{}' = \{x_k | i+1 \leqslant k \leqslant n\}, S_2 = [x_i]_j - S_1, S_2{}' = [x_{i+1}]_j - S_1{}'$.

Denote $|S_1| = r_1$, $|S_1{}'| = r_2$, $|[x_i]_j| = s$, $|[x_{i+1}]_j| = t$. Then it is not difficult to see that $r_1 = \lfloor \frac{n-i}{j} \rfloor + 1, r_2 = \lfloor \frac{n-(i+1)}{j} \rfloor + 1, s = \lfloor \frac{2n-1-i}{j} \rfloor + 1, t = \lfloor \frac{2n-1-(i+1)}{j} \rfloor + 1$.

Notice that

$$\begin{aligned} r_1 - r_2 &= \left\lfloor \frac{n-i}{j} \right\rfloor + 1 - \left(\left\lfloor \frac{n-i-1}{j} \right\rfloor + 1 \right) \\ &= \left\lfloor \frac{n-i}{j} \right\rfloor - \left\lfloor \frac{n-i-1}{j} \right\rfloor \\ &= 0 \text{ or } 1 \end{aligned}$$

and similarly, $s - t = 0$ or 1.

Hence, we have to consider the following two cases.

Case 1. $r_1 = r_2$.

Let

$$S_1 = \{x_{i_1}, x_{i_2}, \ldots, x_{i_{r_1}}\}, \quad S_2 = \{x_{i_{r_1+1}}, x_{i_{r_1+2}}, \ldots, x_{i_s}\}$$

and

$$S_1{}' = \{x_{j_1}, x_{j_2}, \ldots, x_{j_{r_1}}\}, \quad S_2{}' = \{x_{j_{r_1+1}}, x_{j_{r_1+2}}, \ldots, x_{j_t}\},$$

where $i_1 < i_2 < \cdots < i_s$ and $j_1 < j_2 < \cdots < j_t$.

Then

$$\phi(x_{j_1}) - \phi(x_{i_1}) = 1, \quad \phi(x_{j_2}) - \phi(x_{i_2}) = 1, \ldots, \phi(x_{j_{r_1}}) - \phi(x_{i_{r_1}}) = 1 \tag{3.3}$$

and

$$\phi(x_{j_{r_1+1}}) - \phi(x_{i_{r_1+1}}) = -1, \quad \phi(x_{j_{r_1+2}}) - \phi(x_{i_{r_1+2}}) = -1, \ \ldots, \ \phi(x_{j_t}) - \phi(x_{i_t}) = -1. \tag{3.4}$$

Let $\phi(S_i) = \sum_{x_k \in S_i} \phi(x_k)$ and $\phi(S_i{}') = \sum_{x_k \in S_i{}'} \phi(x_k)$, $i = 1, 2$. By (3.3) and (3.4), we have

$$\begin{aligned}
\phi([x_i]_j) - \phi([x_{i+1}]_j) &= \phi(S_1) + \phi(S_2) - [\phi(S_1{}') + \phi(S_2{}')] \\
&= \phi(S_1) - \phi(S_1{}') + \phi(S_2) - \phi(S_2{}') \\
&\geqslant -r_1 + |S_2{}'| \\
&= -r_1 + t - r_2 \\
&= t - 2r_2 \\
&= \left\lfloor \frac{2n-2-i}{j} \right\rfloor + 1 - 2\left(\left\lfloor \frac{n-1-i}{j} \right\rfloor + 1 \right) \\
&= \left\lfloor \frac{2n-2-i+j}{j} \right\rfloor - 2\left\lfloor \frac{n-1-i+j}{j} \right\rfloor \\
&\geqslant \left\lfloor \frac{2n-2-i+j}{j} - 2\frac{n-1-i+j}{j} \right\rfloor \\
&= \left\lfloor \frac{i-j}{j} \right\rfloor.
\end{aligned} \tag{3.5}$$

Case 2. $r_1 = r_2 + 1$.

In this case, we partition S_1 and $S_2{}'$ as

$$S_1 = S_{11} \cup \{x_{i+jr_2}\}, \quad S_2{}' = \{x_{i+1+jr_2}\} \cup S_{22}{}',$$

respectively, where $|S_{11}| = |S_1{}'| = r_2$.

Since $i + jr_2 \leqslant n$ and $i + 1 + jr_2 > n$, it follows that $i + jr_2 = n$ and $i + 1 + jr_2 = n + 1$. In a way similar to that of Case 1, we have

$$\begin{aligned}
\phi([x_i]_j) - \phi([x_{i+1}]_j) &= \phi(S_1) + \phi(S_2) - [\phi(S_1{}') + \phi(S_2{}')] \\
&= \phi(S_{11}) - \phi(S_1{}') + \phi(\{x_{i+jr_2}\} \cup S_2) - \phi(S_2{}') \\
&\geqslant -r_2 + t - r_2 \\
&= t - 2r_2 \\
&\geqslant \left\lfloor \frac{i-j}{j} \right\rfloor.
\end{aligned} \tag{3.6}$$

Now, we can see from (3.5) and (3.6) that if $i \geqslant j$, then $\phi([x_i]_j) \geqslant \phi([x_{i+1}]_j)$. Moreover, if $i \geqslant 2j$, then $\phi([x_i]_j) > \phi([x_{i+1}]_j)$. This completes the proof of (3.1) and (3.2). $\square$

Theorem 13. *Let $A \in S_n(\alpha, \beta)$ be a nonzero symmetric 7-matrix of the form (2.1), $n \geqslant 3$.*
(1) If $x_1 = 0$, then

$$\max \phi(A) = \begin{cases} \frac{n^2-1}{3}, & \text{if } n \equiv 2 \pmod 3, \\ \frac{n^2+2}{3}, & \text{if } n \equiv 1 \pmod 3, \\ \frac{n^2}{3}, & \text{if } n \equiv 0 \pmod 3). \end{cases}$$

Furthermore, the maximum is attained if and only if $\alpha + \beta^2 = 0$.
(2) If $x_2 = 0$, then

$$\max \phi(A) = \begin{cases} \frac{n^2+2}{3}, & \text{if } n \equiv 2 \pmod 3, \\ \frac{n^2-1}{3}, & \text{if } n \equiv 1 \pmod 3, \\ \frac{n^2}{3}, & \text{if } n \equiv 0 \pmod 3). \end{cases}$$

Furthermore, the maximum is attained if and only if $\alpha + \beta^2 = 0$.
(3) If $x_1 \neq 0$ and $x_2 \neq 0$, then

$$\max \phi(A) = \begin{cases} \frac{n^2-1}{3}, & \text{if } n \equiv 2 \pmod 3 \text{ or } n \equiv 1 \pmod 3, \\ \frac{n^2}{3}, & \text{if } n \equiv 0 \pmod 3). \end{cases}$$

Furthermore, the maximum is attained if and only if $x_2 = \beta x_1$ and $\alpha + \beta^2 = 0$.

Proof. (1) We claim that if $x_1 = 0$, then for any integer i, $4 \leqslant i \leqslant 2n-1$, $x_i = 0$ implies $[x_i]_{i-1} = \{0\}$. Indeed, x_i can be expressed as a linear combination of x_1 and x_2, i.e., there exist polynomials $f_1(\alpha, \beta)$ and $f_2(\alpha, \beta)$ such that $0 = x_i = f_1(\alpha, \beta)x_1 + f_2(\alpha, \beta)x_2$. Since A is nonzero and $x_1 = 0$, we have $x_2 \neq 0$ and thus $f_2(\alpha, \beta) = 0$. We now consider $x_{i+(i-1)k}$, $1 \leqslant k \leqslant \lfloor \frac{2n-1-i}{i-1} \rfloor$. If $k = 1$, then $x_{2i-1} = f_1(\alpha, \beta)x_i + f_2(\alpha, \beta)x_{i+1} = 0$. If $k = 2$, then $x_{3i-2} = f_1(\alpha, \beta)x_{2i-1} + f_2(\alpha, \beta)x_{2i} = 0$. Continuing in this way we deduce that $x_{i+(i-1)k} = 0$ for every integer k, $1 \leqslant k \leqslant \lfloor \frac{2n-1-i}{i-1} \rfloor$, i.e., $[x_i]_{i-1} = \{0\}$.

Next, we will prove the following statement.

(S_1) For $4 \leqslant p < q \leqslant 2n-1$, $x_p = x_q = 0$ if and only if there exists some integer w, $4 \leqslant w \leqslant p$, such that $x_w = 0$ and $x_p, x_q \in [x_w]_{w-1}$.

The sufficiency is obvious and we only show the necessity. If $x_q \in [x_p]_{p-1}$, then we are done. If $x_q \notin [x_p]_{p-1}$, then we consider the following two cases.

Case 1. $q \in (p, t)$, where $t = p + (p-1)j_0 \in [x_p]_{p-1}$, $j_0 = \lfloor \frac{2n-1-p}{p-1} \rfloor$.

In this case, there exists some integer j_1, $0 \leqslant j_1 \leqslant j_0 - 1$, such that $q \in (a, b)$, where $a = p + (p-1)j_1 \in [x_p]_{p-1}$, $b = p + (p-1)(j_1+1) \in [x_p]_{p-1}$. If $a < q < \frac{a+b}{2}$, then denote $\delta_0 = q - a$. If $\frac{a+b}{2} \leqslant q < b$, then denote $\gamma_0 = b - q$. Having in mind that A is nonzero, then any three successive x_i's have at most one zero, which implies that $\delta_0 \geqslant 3$ and $\gamma_0 \geqslant 3$.

If $a < q < \frac{a+b}{2}$ and $b - a$ is divisible by δ_0, then $b - a = r\delta_0$ for some integer r. A simple computation shows that $x_a = x_{a+\delta_0} (= x_q) = x_{a+2\delta_0} = \cdots = x_{a+r\delta_0} = x_b = x_{b+\delta_0} = \cdots = x_t = 0$. On the other hand, let $x_{a+\delta_0} = g_1(\alpha, \beta)x_a + g_2(\alpha, \beta)x_{a+1}$ for some polynomials $g_1(\alpha, \beta)$ and $g_2(\alpha, \beta)$. Then $x_{a+\delta_0} = x_a = 0$ implies $g_2(\alpha, \beta) = 0$. Notice that $g_1(\alpha, \beta) \neq 0$. Otherwise we have $x_{a+\delta_0+1} = g_1(\alpha, \beta)x_{a+1} + g_2(\alpha, \beta)x_{a+2} = 0$, contradicting the assumption that A is nonzero. Hence, it follows from $x_a = g_1(\alpha, \beta)x_{a-\delta_0} + g_2(\alpha, \beta)x_{a-\delta_0+1} = 0$ that $x_{a-\delta_0} = 0$. Continuing in this way we have $x_a = x_{a-\delta_0} = \cdots = x_p = x_{p-\delta_0} = \cdots = x_{\delta_0+1} = x_1 = 0$. Hence, $x_p, x_q \in [x_{\delta_0+1}]_{\delta_0}$, $4 \leqslant \delta_0 + 1 \leqslant p$, $x_{\delta_0+1} = 0$ and then the necessity of the statement (S_1) follows. Similarly, the necessity of the statement (S_1) is true if $\frac{a+b}{2} \leqslant q < b$ and $b - a$ is divisible by γ_0.

If $a < q < \frac{a+b}{2}$ and $b - a$ is not divisible by δ_0, then there exist unique integers q_1 and $\delta_1 (< \delta_0)$ such that $b - a = q_1\delta_0 + \delta_1$. Hence, $x_a = x_{a+\delta_0} (= x_q) = x_{a+2\delta_0} = \cdots = x_{b-\delta_1} = x_b = 0$. If $b - a$ is not divisible by δ_1, then there exist unique integers q_2 and $\delta_2 (< \delta_1)$ such that $b - a = q_2\delta_1 + \delta_2$. Hence,

$x_b = x_{b-\delta_1} = \cdots = x_{a+\delta_2} = x_a = 0$. Continuing in this way we conclude that $b - a = q_1\delta_0 + \delta_1 = q_2\delta_1 + \delta_2 = \cdots = q_i\delta_{i-1} + \delta_i = q_{i+1}\delta_i$, i.e., there exists some integer $\delta_i(\geqslant 3)$ such that δ_i divides $b-a$. Otherwise we will obtain a strictly decreasing sequence $\delta_0, \delta_1, \cdots, \delta_n$ such that $\delta_n \leqslant 2$, which implies that $x_a = x_{a+\delta_n} = 0$ or $x_{b-\delta_n} = x_b = 0$, contradicting the assumption that A is nonzero. Now, assume that $q \in [a^{(1)}, b^{(1)}]$, where $a^{(1)} = a + l\delta_i$, $b^{(1)} = a + (l+1)\delta_i$, $0 \leqslant l \leqslant q_{i+1} - 1$. Then $b^{(1)} - a^{(1)}$ divides $b - a$. If $q = a^{(1)}$ or $q = b^{(1)}$, then by the above argument, the necessity of the statement (S_1) follows. If $q \in (a^{(1)}, b^{(1)})$ and $q - a^{(1)}$ or $b^{(1)} - q$ divides $b^{(1)} - a^{(1)}$, then by the above argument, the necessity of the statement (S_1) also follows. If $b^{(1)} - a^{(1)}$ is not divisible by $q - a^{(1)}$ or $b^{(1)} - q$, then by the above argument, we will find a smaller interval $[a^{(2)}, b^{(2)}] \subseteq [a^{(1)}, b^{(1)}]$ such that $b^{(2)} - a^{(2)}$ divides $b^{(1)} - a^{(1)}$ and $q \in [a^{(2)}, b^{(2)}]$. Repeating the above argument and noting that $[a, b]$ is a finite interval, we conclude that there exists some interval $[a^{(m)}, b^{(m)}] \subseteq [a^{(m-1)}, b^{(m-1)}] \subseteq \cdots \subseteq [a, b]$ such that $b^{(m)} - a^{(m)}$ divides $b - a$ and $q \in [a^{(m)}, b^{(m)}]$, either $q = a^{(m)}$ or $q = b^{(m)}$, or, either $q - a^{(m)}$ or $b^{(m)} - q$ divides $b^{(m)} - a^{(m)}$. Therefore, if $x_p = x_q = 0$ and $b - a$ is not divisible by δ_0 or γ_0, then there exists some integer $\mu(\geqslant 3)$ such that μ divides $b - a$ and $q = a + k\mu$ for some positive integer k, i.e., $x_p, x_q \in [x_{\mu+1}]_\mu$, $4 \leqslant \mu + 1 \leqslant p$ and $x_{\mu+1} = 0$. Hence, the necessity of the statement (S_1) is true.

Case 2. $q \in (t, 2n-1]$.

Let $\theta = p + (p-1)(j_0 - 1)$. Then $t \in (\theta, q)$. If $q - t$ divides $q - \theta$, then $q - t$ divides $t - \theta$. Hence, $x_q = x_t = x_{t-(q-t)} = \cdots = x_\theta = x_{\theta-(q-t)} = \cdots = x_p = x_{p-(q-t)} = \cdots = x_{q-t+1} = x_1 = 0$ and then the necessity of the statement (S_1) follows. If $q - t$ is not divisible by $q - \theta$, then by the above argument, there exists some integer $\lambda(\geqslant 3)$ such that λ divides $q - \theta$ and $t = \theta + l\lambda$ for some integer l. Thus, λ divides $t - \theta$ and λ also divides $q - t$. Hence, $x_q = x_{q-\lambda} = \cdots = x_t = x_{t-\lambda} = \cdots = x_\theta = x_{\theta-\lambda} = \cdots = x_p = x_{p-\lambda} = \cdots = x_{\lambda+1} = x_1 = 0$ and then $x_p, x_q \in [x_{\lambda+1}]_\lambda$, $4 \leqslant \lambda + 1 \leqslant p$, $x_{\lambda+1} = 0$. This completes the proof of the statement (S_1).

Note that the statement (S_1) implies

$$\max\{\phi(A) | A = [0, x_2]_{\alpha,\beta} \in S_n(\alpha, \beta), x_2 \neq 0\} = \max_{4 \leqslant i \leqslant n} \phi([x_i]_{i-1}) + 1.$$

Now, we are left with determining $\max_{4\leqslant i\leqslant n} \phi([x_i]_{i-1})$. To this end, we will prove that

$$\phi([x_i]_{i-1}) > \phi([x_{i+1}]_i) \text{ for any integer } i,\ 4 \leqslant i \leqslant 2n-2. \tag{3.7}$$

That is, $\phi([x_4]_3) = \max_{4\leqslant i\leqslant n} \phi([x_i]_{i-1})$.

If $n \leqslant i \leqslant 2n-2$, then it is easy to check that $\phi([x_i]_{i-1}) > \phi([x_{i+1}]_i)$.

If $4 \leqslant i \leqslant n-1$, let $[x_{i+1}]_{i-1} = \{x_{i+1}\} \cup [x_{2i}]_{i-1}$. Then by Lemma 12,

$$\phi([x_i]_{i-1}) \geqslant \phi([x_{i+1}]_{i-1}) = \phi(x_{i+1}) + \phi([x_{2i}]_{i-1}) > \phi(x_{i+1}) + \phi([x_{2i+1}]_{i-1}).$$

Let $[x_{2i+1}]_{i-1} = \{x_{2i+1}\} \cup [x_{3i}]_{i-1}$. Then

$$\begin{aligned}\phi([x_i]_{i-1}) &> \phi(x_{i+1}) + \phi([x_{2i+1}]_{i-1})\\ &= \phi(x_{i+1}) + \phi(x_{2i+1}) + \phi([x_{3i}]_{i-1})\\ &> \phi(x_{i+1}) + \phi(x_{2i+1}) + \phi([x_{3i+1}]_{i-1}).\end{aligned}$$

Continuing in this way we finally get

$$\phi([x_i]_{i-1}) > \phi(x_{i+1}) + \phi(x_{2i+1}) + \phi(x_{3i+1}) + \cdots + \phi(x_{i+1+i\lfloor\frac{2n-2-i}{i}\rfloor}) = \phi([x_{i+1}]_i),$$

which completes the proof of (3.7).

Hence, if $n \geqslant 3$, then $\max \phi(A) = \phi([x_4]_3) + 1$. Now, it remains only to compute $\phi([x_4]_3) + 1$.

If $n \equiv 2 \pmod 3$, then $\phi([x_4]_3) + 1 = \sum_{j=1}^{\frac{n+1}{3}} (3j-2) + \sum_{j=1}^{\frac{n-2}{3}} [n - (3j-1)] = \frac{n^2-1}{3}$.

If $n \equiv 1 \pmod 3$, then $\phi([x_4]_3) + 1 = 2\sum_{j=1}^{\frac{n-1}{3}}(3j-2) + n = \frac{n^2+2}{3}$.

If $n \equiv 0 \pmod 3$, then $\phi([x_4]_3) + 1 = \sum_{j=1}^{\frac{n}{3}}(3j-2) + \sum_{j=1}^{\frac{n}{3}}[n-(3j-2)] = \sum_{j=1}^{\frac{n}{3}} n = \frac{n^2}{3}$.

Furthermore, the above maximum is attained if and only if $x_4 = 0$, i.e., $\alpha + \beta^2 = 0$.

(2) If $x_2 = 0$, let us partition A as

$$A = \begin{pmatrix} y & A_{n-1} \\ x_n & z^T \end{pmatrix},$$

where $y, z \in \mathbb{R}^{n-1}$, $A_{n-1} = [x_2, x_3]_{\alpha,\beta} \in S_{n-1}(\alpha, \beta)$.

Let $w = (y^T, x_n, z^T)^T = (x_1, x_2, \ldots, x_{2n-1})^T \in V_{2n-1}(\alpha, \beta)$ and let $\phi(w)$ be the number of zero components of w. Then $\phi(A) = \phi(A_{n-1}) + \phi(w)$. It is easy to see that

$$\max_{\substack{w \in V_{2n-1}(\alpha,\beta) \\ x_2=0}} \phi(w) = \left\lfloor \frac{2n-1-2}{3} \right\rfloor + 1 = \left\lfloor \frac{2}{3}n \right\rfloor \tag{3.8}$$

and $\phi(w) = \lfloor \frac{2}{3}n \rfloor$ if and only if $x_5 = 0$. On the other hand, by the case (1) above,

$$\max \phi(A_{n-1}) = \begin{cases} \frac{(n-1)^2+2}{3}, & \text{if } n \equiv 2 \pmod 3, \\ \frac{(n-1)^2}{3}, & \text{if } n \equiv 1 \pmod 3, \\ \frac{(n-1)^2-1}{3}, & \text{if } n \equiv 0 \pmod 3). \end{cases} \tag{3.9}$$

and the maximum in (3.9) is attained if and only if $x_5 = 0$.

Hence, by (3.8) and (3.9), we have

$$\max \phi(A) = \begin{cases} \frac{n^2+2}{3}, & \text{if } n \equiv 2 \pmod 3, \\ \frac{n^2-1}{3}, & \text{if } n \equiv 1 \pmod 3, \\ \frac{n^2}{3}, & \text{if } n \equiv 0 \pmod 3). \end{cases} \tag{3.10}$$

Furthermore, the maximum in (3.10) is attained if and only if $x_5 = 0$, i.e., $\alpha + \beta^2 = 0$.

(3) Let

$$\Theta = \max\{\phi(A) \mid A = [x_1, x_2]_{\alpha,\beta} \in S_n(\alpha, \beta), x_1 \neq 0, x_2 \neq 0\}.$$

Then

$$\Theta = \max_{3 \leqslant i \leqslant 2n-1} \max_{\substack{x_i=0 \\ x_j \neq 0, 1 \leqslant j < i}} \phi(A). \tag{3.11}$$

To determine (3.11), we first determine $\max_{\substack{x_i=0 \\ x_j \neq 0, 1 \leqslant j < i}} \phi(A)$. If $x_i = 0$ for some integer i, $3 \leqslant i \leqslant 2n-1$, and $x_j \neq 0$ for every j, $1 \leqslant j < i$, then $x_{i+k} \neq 0$ for every k, $1 \leqslant k \leqslant i-1$. Otherwise by $x_i = x_{i+k} = 0$, $1 \leqslant k \leqslant i-1$, we have $x_{i-k} = 0$, contradicting our assumption that $x_j \neq 0$ for every j, $1 \leqslant j < i$. Hence, if $x_i = 0$ for some i, $n \leqslant i \leqslant 2n-1$ and $x_j \neq 0$ for every j, $1 \leqslant j < i$, then $x_k \neq 0$ for every k, $k \in \{1, 2, \ldots, 2n-1\} \backslash \{i\}$. However, if $3 \leqslant i \leqslant n-1$, it may happen that $x_{i+m} = 0$ for some integer m, $i \leqslant m \leqslant 2n-1-i$.

The following statement plays an important role in determining $\max_{\substack{x_i=0 \\ x_j \neq 0, 1 \leqslant j < i}} \phi(A)$, $3 \leqslant i \leqslant n-1$.

(S_2) For three integers i, r and s satisfying $3 \leqslant i \leqslant n-1$, $i \leqslant s \leqslant 2n-1-i$ and $i \leqslant r \leqslant 2n-1$, if $[x_i]_s = \{0\}$, $x_r = 0$ and $x_r \notin [x_i]_s$, then there exists some integer $t < s$, such that $[x_i]_s \subseteq [x_i]_t$, $x_r \in [x_i]_t$ and $[x_i]_t = \{0\}$.

The proof of the statement (S_2) is almost the same as that of the necessity of the statement (S_1) and we will omit it.

We remark that the statement (S_2) implies that if $3 \leqslant i \leqslant n-1$, then $\max_{\substack{x_i=0 \\ x_j \neq 0, 1 \leqslant j < i}} \phi(A) = \phi([x_i]_c)$ for some integer c, $i \leqslant c \leqslant 2n-1-i$.

Next, we will show that

$$\phi([x_i]_c) > \phi([x_i]_{c+1}) \text{ for each pair of integers } i,\ c,\ 3 \leqslant i \leqslant n-1,\ i \leqslant c \leqslant 2n-1-i, \quad (3.12)$$

which means that if $3 \leqslant i \leqslant n-1$, then $\max_{\substack{x_i=0 \\ x_j \neq 0, 1 \leqslant j < i}} \phi(A) = \phi([x_i]_i)$.

If $3 \leqslant i \leqslant n-1$, by Lemma 12, we have

$$\begin{aligned}
\phi([x_i]_c) &= \phi(x_i) + \phi([x_{i+c}]_c) \\
&\geqslant \phi(x_i) + \phi([x_{i+c+1}]_c) \\
&= \phi(x_i) + \phi(x_{i+c+1}) + \phi([x_{i+2c+1}]_c) \\
&> \phi(x_i) + \phi(x_{i+c+1}) + \phi([x_{i+2c+2}]_c) \\
&= \phi(x_i) + \phi(x_{i+c+1}) + \phi(x_{i+2c+2}) + \phi([x_{i+3c+2}]_c) \\
&> \phi(x_i) + \phi(x_{i+c+1}) + \phi(x_{i+2c+2}) + \phi(x_{i+3c+3}) + \cdots \\
&= \phi([x_i]_{c+1}),
\end{aligned}$$

which completes the proof of (3.12).

If $n \leqslant i \leqslant 2n-1$, then $\max_{\substack{x_i=0 \\ x_j \neq 0, 1 \leqslant j < i}} \phi(A) = \phi(x_i) = 2n-i$. It is not difficult to see that

$$\max_{\substack{x_{n-1}=0 \\ x_j \neq 0, 1 \leqslant j < n-1}} \phi(A) > \max_{\substack{x_n=0 \\ x_j \neq 0, 1 \leqslant j < n}} \phi(A) > \cdots > \max_{\substack{x_{2n-1}=0 \\ x_j \neq 0, 1 \leqslant j < 2n-1}} \phi(A).$$

Thus, $\Theta = \max_{3 \leqslant i \leqslant 2n-1} \max_{\substack{x_i=0 \\ x_j \neq 0, 1 \leqslant j < i}} \phi(A) = \max_{3 \leqslant i \leqslant n-1} \max_{\substack{x_i=0 \\ x_j \neq 0, 1 \leqslant j < i}} \phi(A) = \max_{3 \leqslant i \leqslant n-1} \phi([x_i]_i)$.

If $3 \leqslant i \leqslant n-1$, by Lemma 12 we have

$$\begin{aligned}
\phi([x_i]_i) &\geqslant \phi([x_{i+1}]_i) \\
&= \phi(x_{i+1}) + \phi([x_{2i+1}]_i) \\
&> \phi(x_{i+1}) + \phi([x_{2i+2}]_i) \\
&= \phi(x_{i+1}) + \phi(x_{2i+2}) + \phi([x_{3i+2}]_i) \\
&> \phi(x_{i+1}) + \phi(x_{2i+2}) + \phi(x_{3i+3}) + \cdots \\
&= \phi([x_{i+1}]_{i+1}).
\end{aligned}$$

Therefore, we have proven that $\Theta = \phi([x_3]_3)$. Now, it remains only to compute $\phi([x_3]_3)$.

Let us partition A as

$$A = \begin{pmatrix} x_1 & v^T \\ v & A_1 \end{pmatrix},$$

where $A_1 = [x_3, x_4]_{\alpha,\beta} \in S_{n-1}(\alpha, \beta)$.

Hence, $\Theta = \phi([x_3]_3) = \max_{x_3=0} \phi(A_1) + 2\max_{x_3=0} \phi(v)$.

By the case (1), we have

$$\max_{x_3=0} \phi(A_1) = \begin{cases} \frac{(n-1)^2+2}{3}, & \text{if } n \equiv 2 \pmod 3, \\ \frac{(n-1)^2}{3}, & \text{if } n \equiv 1 \pmod 3, \\ \frac{(n-1)^2-1}{3}, & \text{if } n \equiv 0 \pmod 3. \end{cases} \tag{3.13}$$

The maximum in (3.13) is attained if and only if $x_6 = 0$.

It is not difficult to see that $2\max_{x_3=0} \phi(v) = 2\lfloor \frac{n}{3} \rfloor$ and $\phi(v) = \lfloor \frac{n}{3} \rfloor$ if and only if $x_6 = 0$. Therefore,

$$\Theta = \phi([x_3]_3) = \begin{cases} \frac{n^2-1}{3}, & \text{if } n \equiv 2 \pmod 3 \text{ or } n \equiv 1 \pmod 3, \\ \frac{n^2}{3}, & \text{if } n \equiv 0 \pmod 3. \end{cases} \tag{3.14}$$

Furthermore, the maximum in (3.14) is attained if and only if $[x_3]_3 = \{0\}$, which is equivalent to $x_3 = x_6 = 0$, i.e., $x_2 = \beta x_1$ and $\alpha + \beta^2 = 0$. $\square$

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